2 *The Lorentz Transformation*

§2-1 **The Lorentz Transformation** We may consider a coordinate transformation between two inertial frames as a dictionary by whose means observers residing on the two frames can make their meaning clear to each other. In ordinary experience we need the assistance of the Galilean transformation to reconcile the determination of velocities on two inertial frames. An observer on the ground who measures the velocity of an airplane with respect to his frame will arrive at an entirely different result than the pilot reading his airspeed indicator and his compass heading. If these observers communicate by radio, their results are completely contradictory until the conflict is resolved through the Galilean transformation.

Consider the requirements upon a set of transformation equations which will fulfill our needs:

(1) The transformation must be linear; that is, a single event in one inertial frame must transform to a single event in another frame, with a single set of coordinates.

(2) The transformation must approach the Galilean transformation in the limit of low speeds. Here low speeds mean low compared to c, so that we wish to examine the transformation in the limit $\beta \rightarrow 0$.

(3) The speed of light must have the same value, c, in every inertial frame.

Just as the disturbance in a pond resulting from dropping a pebble into the water is a system of circular ripples, so a flash of light spreads out as a growing sphere. We may describe this sphere, whose radius grows at speed c, by the equation

$$x^2 + y^2 + z^2 = c^2t^2. \qquad (2\text{-}1.1)$$

Just as the spreading light flash forms a spreading sphere in the unprimed frame, so it must also form a spreading sphere in the primed frame. Suppose the flash of light takes place at the instant when both t and t' are zero, and when the origins of the two frames coincide. Then the equation of the sphere of light in the primed frame must be

$$x'^2 + y'^2 + z'^2 = c^2t'^2. \tag{2-1.2}$$

The coordinate transformation that satisfies these requirements is called the Lorentz transformation, after its discoverer, and forms the mathematical beginning of the special theory of relativity. The space and time coordinates of an event recorded in two inertial frames whose axes are parallel, and whose origins coincide at time zero (in both frames), are related by the equations

$$\begin{aligned} x' &= \gamma(x - Vt), &\quad \text{and conversely}\quad x &= \gamma(x' + Vt'); &\quad (2\text{-}1.3)\\ y' &= y, & y &= y';\\ z' &= z, & z &= z';\\ t' &= \gamma(t - Vx/c^2), & t &= \gamma(t' + Vx'/c^2). \end{aligned}$$

In these equations the primed frame is taken to be moving with speed V in the $+x$ direction with respect to the unprimed frame. Conversely the unprimed frame may be taken to be moving with speed V in the $-x'$ direction with respect to the primed frame. As indicated in Eqs. 1-5.2, we use the symbol $\gamma = (1 - V^2/c^2)^{-1/2}$.

The Lorentz transformation fulfills the first two requirements we have stated. It is linear, and in the limit $V \to 0$, $\gamma \to 1$, the Eqs. 2-1.3 become the Galilean transformation, Eqs. 1-2.1; the Lorentz transformation yields the results of ordinary experience in the limit of low velocities.

By direct substitution of Eqs. 2-1.3 into the equation of the light sphere in the unprimed frame, Eq. 2-1.1, we find the equation of the light sphere in the primed frame.

Thus the Lorentz transformation fulfills requirement (3) above. A pulse of light emitted at the coincident origin of the

two frames spreads out as a sphere in both frames. When observers in the unprimed frame translate their observations to the language of the primed frame, the translated observation is in agreement with observations of the primed observers, that a light sphere spreads out from the origin of the primed coordinates with speed c.

2-1.1 By direct application of the Lorentz transformation equations, show that Eq. 2-1.1 is transformed into Eq. 2-1.2.

2-1.2 By taking differentials of the Lorentz transformation equations, show that the quantity ds transforms to ds', where $ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$, and $ds'^2 = dx'^2 + dy'^2 + dz'^2 - c^2dt'^2$.

§2-2 Simultaneity and Time Sequence Suppose we now examine some implications of the Lorentz transformations. Let there be as many observers in each inertial frame as are required. Observers in each frame have access to identical meter sticks, and to identical clocks. They use identical procedures in their measurements. Each measured event is described at least by a set of four coordinates, giving the position and time at which the event took place.

In any inertial frame the clocks of all observers are synchronized. They have been checked by stationing a third observer midway between any two. He notes that the pair of clocks reads the same. This procedure takes account of the finite speed of light, and insures that there are no alterations in the clock's behavior, as there might be if they were carried to a central timing station.

In any one frame there is no doubt as to when two events are simultaneous, wherever they occur. If two events occur at the same time, the same reading of the clocks located at the places where the events occurred assures us that the events were simultaneous.

Let us consider two events, 1 and 2, which are simultaneous in the unprimed frame. Then $t_1 = t_2$, wherever the events took place. But whether they are noted as simultaneous in the primed frame depends on where the events took place. From the Lorentz transformation, Eqs. 3-1.3, we note that

$$t'_1 = \gamma(t_1 - Vx_1/c^2) \quad \text{and} \quad t'_2 = \gamma(t_2 - Vx_2/c^2).$$

The difference between the time of the two events in the primed frame is

$$t'_2 - t'_1 = \frac{\gamma V}{c^2}(x_1 - x_2). \tag{2-2.1}$$

Two events simultaneous in the unprimed frame are observed to be simultaneous in the primed frame only if they occur at the same point, $x_1 = x_2$. The second event might appear earlier, or later than the first, depending on the value of their x coordinates.

Suppose now that the event t_1 was first, and t_2 was second, in the unprimed frame. The time difference between them may be obtained from the above equations as

$$t'_2 - t'_1 = \gamma\left[(t_2 - t_1) - \frac{V}{c^2}(x_2 - x_1)\right]. \tag{2-2.2}$$

If the quantity in brackets is equal to zero, then the two events are simultaneous in the primed frame. If greater than zero, then the events are observed in the primed frame in the same order as in the unprimed frame. If less than zero, then the events are observed in reverse order. The last case can arise only if

$$x_2 - x_1 > c(t_2 - t_1),$$

that is, if the two events happen at such remote places that a ray of light leaving event 1 could not have reached point 2 in time to cause event 2. We say that the time sequence of two events can only be inverted if they could not have been "causally connected"; that is, if event 1 could conceivably have caused event 2 (by sending a radio signal, or by tripping a switch), then the order of the events cannot be inverted. No signal can travel faster than the speed of light, so that if a light beam could not bridge the interval between the two events the second event could not have had knowledge of the prior occurrence of the first.

2-2.1 Two light bulbs in the laboratory, one at $x = 0$ and the other at $x = 10$ km, are set to flash simultaneously at $t = 0$. Observers on a magic carpet moving in the $+x$ direction with speed 3×10^7 m/sec observe the flashes. (a) What time interval do they note between flashes?

(b) Which bulb do they say goes off first? [(a) 3.34×10^{-9} sec (b) the bulb at 10 km]

2-2.2 A long straight rod is inclined at angle θ to the x axis. The rod moves in the y direction with velocity V. (a) Find the velocity v with which the point of intersection of the rod and the x-axis moves along the x axis. (b) If $V = 10^{10}$ cm/sec, and $\theta = 0.10$ rad, what is the numerical value of v? (c) Does your result contradict the relativistic demand that c is a limiting speed for all material particles? [(a) $v = V \cot \theta$ (b) 10^{11} cm/sec (c) no, for the point of intersection of the ruler with the x axis is a mathematical point rather than a material particle.]

§2-3 **Time Dilation** Suppose we examine, from the vantage point of the laboratory (the unprimed frame), a clock which is stationary in the primed frame. The primed frame is the proper frame of the clock, for this is the frame in which the clock is at rest. The clock then is moving with speed V in the $+x$ direction with respect to the laboratory.

Since the clock is in fixed x' position, the time interval between two beats of the clock at t'_1 and t'_2 will appear in the laboratory frame to be

$$t_2 - t_1 = \gamma(t'_2 - t'_1), \tag{2-3.1}$$

by application of the Lorentz transformation, Eqs. 2-1.3, for $x'_2 = x'_1$. We may therefore set $x'_2 = x'_1 = 0$ if we wish, for nothing is changed if the clock is at the origin of the primed coordinates.

This is the same result that we have obtained earlier for the rod clock, §1-5, in a slightly different notation. But here the result does not depend on the detailed design of the clock.

One variety of subatomic particles, called mesons, decays at an exponential rate such that $1/e$ of the original number remains after 2.6×10^{-8} sec, in a coordinate frame in which the mesons are at rest. Beams of π mesons may be produced by bombarding the target of an accelerator with high energy protons; the resulting mesons move out of the target at a speed which can be $0.99c$. In their proper frame π mesons decay at their proper rate, but laboratory observers measuring the decay rate of the moving mesons must find that the meson clocks run slow. From Eq. 2-3.1

the decay time will be measured in the laboratory to be $2.6 \times 10^{-8} \times (1 - 0.99^2)^{-1/2}$ sec, or 1.87×10^{-7} sec. In that time a meson beam moving with a speed of nearly 3×10^8 m/sec will move a distance of 56 m. Thus in 56 m the meson beam will drop to $1/e$ of its initial intensity. If this relativistic effect did not exist, we would expect the intensity of the beam to drop by $1/e$ each 2.6×10^{-8} sec in the laboratory, regardless of the motion. Since in this case $\gamma = 7.18$, we would here have expected the beam to drop by $e^{-7.18}$, or to drop to 1/1300 of its initial intensity in 56 m. The difference between 1/1300 and $1/e = 1/2.78$ is readily detected. Time dilation is thus confirmed in the laboratory, and is today routinely taken for granted in the design of experiments which must be performed with particle beams in high-energy accelerators.

Let us repeat once again what we mean by time dilation. We mean that a proper time interval is dilated or expanded when measured from some inertial frame other than the proper frame. The proper frame is the frame in which the events took place at a single point. There is no change in any proper frame by virtue of its motion with respect to another frame. In fact it is one of the basic principles of the special theory of relativity that there is no physical effect which can be associated with the motion of an inertial frame. Such a motion cannot be detected by measurements wholly within that frame. It is impossible to build a completely self-contained speedometer which measures the motion of an inertial frame, except by reference to a second frame.

2-3.1 A particle with a mean proper lifetime of 1 μsec moves through the laboratory at 2.7×10^{10} cm/sec. (a) What is its lifetime, as measured by observers in the laboratory? (b) If it was manufactured in the target of an accelerator, how far does it go, on the average, in the laboratory before disintegrating? (c) Repeat the calculation of the preceding part without taking relativity into account. [(a) 2.3 μsec (b) 6.2×10^4 cm (c) 2.7×10^4 cm]

§2-4 Lorentz Contraction Let us set up an imaginary experiment to measure the length of a fixed rod and the length of a moving rod. There is no problem in measuring the length of a

rod which is at rest in a coordinate frame. Observers in the frame simply note the coordinates of the ends of the rod, at their convenience, and apply the Pythagorean theorem to determine the length. Or, more simply, they might line up the rod with the x axis, and take the difference in the x coordinates of the ends of the rod as the length. But how do we measure the length of a moving rod?

Suppose we put observers everywhere along the x axis in the laboratory frame when the rod is moving in the $+x$ direction with speed V and is aligned parallel to the x axis. We ask the observers to synchronize their watches, and then to give some signal if the end of the rod is at their coordinate position at a predetermined time. The difference in the coordinates of the two observers in the laboratory frame who signal that they see opposite ends of the rod at the appointed time is to be taken as the length of the rod.

The results of the length measurement clearly depend on the meaning of simultaneity. Depending on the choice of the recipe for synchronizing watches, the rod could turn out to have any length at all. The relativistic recipe has already been chosen, and has been incorporated into the Lorentz transformation. We compare the coordinates of the ends of the rod in the primed and unprimed frames at the same laboratory (unprimed) time. From Eqs. 2-1.3 we find

$$x'_2 = \gamma(x_2 - Vt) \quad \text{and} \quad x'_1 = \gamma(x_1 - Vt),$$

which we subtract:

$$x'_2 - x'_1 = \gamma(x_2 - x_1). \tag{2-4.1}$$

On the left of Eq. 2-4.1 we have the difference in the coordinates of the ends of the rod as determined in the primed frame, the proper frame of the rod. On the right the difference in the coordinates of the ends of the rod as determined in the laboratory frame appear. Let us make the identifications

$$x'_2 - x'_1 = L' \quad \text{and} \quad x_2 - x_1 = L,$$

so that we have

$$L' = \gamma L. \tag{2-4.2}$$

Remembering that γ is at least 1, we find that the measured length of the rod in the laboratory frame, L, is less than its proper length L' by the factor γ. This effect is today called the *Lorentz contraction,* for the formula with which Lorentz attempted to explain the negative result of the Michelson-Morley experiment is precisely the same as we have found here, though the interpretation of the formula is now different from his.

Notice that we do not require that rods shrink in their direction of motion for a proper observer, for whom the rod is at rest, as Lorentz proposed. Relativity requires that the contraction is related to the act of measurement. The way in which we set up to measure the length of a moving rod determines that we will measure a shorter length than the rod length. We would not have come to such a conclusion if the speed of light were infinite, for then β would be zero and the Lorentz transformation would be identical to the Galilean transformation. It is the finite speed of light which gives rise to these results.

Is the moving rod *really* contracted in its direction of motion? Is time *really* dilated? These questions depend on what is meant by *really*. In physics what is real is identical with what is measured. There is no way to assign properties to a rod or to a clock or to an electron except through measurement. In this sense the phenomena we have discussed as time dilation and the Lorentz contraction are real. But a physicist tries to unite his observations into a concise set of ideas. He measures the lifetime of π mesons at rest and in motion and finds it appropriate to combine these measurements into a *proper lifetime,* for all other lifetimes are easily calculable from this one. In general it is the proper length or the proper lifetime which one assigns to the meson or the rod. But in the laboratory moving mesons are *really* alive long after their twins, born at the same time, and at rest, are *really* dead.

2-4.1 A rod has a length of 1 m. When the rod is in a satellite moving with respect to the earth at $0.99c$, what is the length of the rod as determined by an observer in the satellite? [1 m]

2-4.2 A vector represented in coordinate form in the primed frame is given as $8\ \mathbf{1}_{x'} + 6\ \mathbf{1}_{y'}$. Find its representation in the unprimed frame, if the primed frame moves at $\mathbf{V} = 0.75c\ \mathbf{1}_x$ with respect to the unprimed frame. [$5.3\ \mathbf{1}_x + 6\ \mathbf{1}_y$]

2-4.3 A thin rod of length L_0 when measured by a proper observer is moving at $0.75c$ with respect to a second observer in a direction at 37° to its own length. What is the length L of the rod as measured by the second observer and his assistants? [$0.8\ L_0$]

§2-5 Velocity Transformations To find the Lorentz transformations of the velocities we employ the same technique we have used before, in the case of the Galilean transformation, §1-2. By taking differentials of Eqs. 2-1.3 we find

$$dx' = \gamma(dx - V\,dt), \quad \text{and} \quad dx = \gamma(dx' + V\,dt'); \tag{2-5.1a}$$

$$dy' = dy, \qquad dy = dy'; \tag{2-5.1b}$$

$$dz' = dz, \qquad dz = dz'; \tag{2-5.1c}$$

$$dt' = \gamma(dt - V\,dx/c^2), \qquad dt' = \gamma(dt' + V\,dx'/c^2). \tag{2-5.1d}$$

When we divide each of the first three equations in each column by the fourth equation of that column, we find

$$U'_x = \frac{U_x - V}{1 - \dfrac{U_x V}{c^2}}, \quad \text{and} \quad U_x = \frac{U'_x + V}{1 + \dfrac{U'_x V}{c^2}}; \tag{2-5.2a}$$

$$U'_{y,z} = \frac{U_{y,z}}{\gamma\left(1 - \dfrac{U_x V}{c^2}\right)}, \qquad U_{y,z} = \frac{U'_{y,z}}{\gamma\left(1 + \dfrac{U'_x V}{c^2}\right)}. \tag{2-5.2b}$$

In these equations we have written $U_x = \dfrac{dx}{dt}$, $U'_x = \dfrac{dx'}{dt'}$, and so on. U_x is the x component of the velocity of the particle as measured in the unprimed frame. U'_x is the x' component of the velocity of the particle as measured in the primed frame. The subscript y,z refers to either the y or the z component. In Eqs. 2-5.2 the primed frame is moving in the $+x$ direction with respect to the unprimed frame with speed V.

Note that Eqs. 2-5.2 reduce to the Galilean velocity transforma-

tions of Eqs. 1-2.3 in the nonrelativistic limit $\beta \to 0$. But there is a significant difference at large velocities. If the moving object is a photon moving in the primed frame with speed c in the $+x$ direction, and if the primed frame moves with speed c with respect to the unprimed frame, then

$$U_x = \frac{c + c}{1 + \frac{c^2}{c^2}} = c.$$

The speed of light is c in all inertial frames, whatever their relative speed. By the Galilean transformation we would have expected $U_x = 2c$, but we know that this is not applicable. Regardless of the value of V, if $U'_x = c$, then $U_x = c$. Of course this is how it must be, for the condition that the speed of light is c in all inertial frames is built into the Lorentz transformation.

2-5.1 (a) Two particles come toward each other, each with speed $0.9c$ with respect to the laboratory. What is their relative speed? (b) Two particles are emitted from a disintegrating source, each moving with speed $0.9c$ with respect to the source. What is their speed relative to each other? [(a) and (b) $0.995c$]

2-5.2 A particle has velocity $\mathbf{U}' = 3\ \mathbf{1}_{x'} + 4\ \mathbf{1}_{y'} + 12\ \mathbf{1}_{z'}$ m/sec in a coordinate frame which itself moves in the $+x$ direction with respect to the laboratory at $V = 0.8c$. Find $\mathbf{U}$ in the laboratory frame. [$\mathbf{U} = 2.4 \times 10^8 \mathbf{1}_x + 2.4\ \mathbf{1}_y + 7.2\ \mathbf{1}_z)$ m/sec]

§2-6 The Fizeau Experiment How can we detect evidence of the velocity transformation formulas in the laboratory? We have already seen that the speed of light in vacuum is c in every inertial frame, but what about its speed in a medium, such as water? The speed of light in a medium must clearly be with respect to a coordinate frame fixed in the medium, for the very structure of the medium, the position of its atoms and molecules, provides a preferred reference frame. The speed of light in a medium is less than c, and the index of refraction for a particular wavelength is $n = c/v$, a number generally greater than one.

If the medium is in motion with respect to the laboratory, at speed V, then the speed of light with respect to the laboratory U